Proceedings of the Jangjeon Mathematical Society
20 (2017), No. 4. pp. 659 – 670

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2017.20.4.659

SOME FORMULAS OF ORDERED BELL NUMBERS AND POLYNOMIALS ARISING FROM UMBRAL CALCULUS

TAEKYUN KIM, DAE SAN KIM, AND GWAN-WOO JANG

Abstract. In this paper, we study the ordered Bell numbers and polynomials and investigate some interesting properties of these numbers and polynomials arising from umbral calculus. In particular, we derive several formulas for the ordered Bell numbers and polynomials, and express well known family of polynomials in terms of the ordered Bell polynomials.

1. Introduction

As is well known, the (unordered) *Bell polynomials* (also called the exponential polynomials) are defined by the generating function

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!}, \text{ (see [7, 10]).} \tag{1.1}$$

From (1.1), we note that

$$Bel_n(x) = \sum_{l=0}^{n} S_2(n,l)x^l, \ (n \geq 0), \tag{1.2}$$

where $S_2(n,l)$ are the Stirling numbers of the second kind.

It is not difficult to show that

$$Bel_n(x) = e^{-x}\sum_{l=0}^{\infty}\frac{l^n}{l!}x^l, \ (n \geq 0).$$

The *Bernoulli polynomials* are given by the generating function

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \text{ (see [1-18]).} \tag{1.3}$$

When $x = 0$, $B_n = B_n(0)$, $(n \geq 0)$ are called the *Bernoulli numbers*.

For $u \in \mathbb{C}$ with $u \neq 1$, the *Frobenius-Euler polynomials* are also defined by the generating function

$$\frac{1-u}{e^t-u}e^{xt} = \sum_{n=0}^{\infty} H_n(u|x)\frac{t^n}{n!}, \text{ (see [11]).} \tag{1.4}$$

When $x = 0$, $H_n(u|0) = H_n(u)$ are called the *Frobenius-Euler numbers*.

In [9], it is known that the *Changhee polynomials* are given by

$$\frac{2}{2+t}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!}. \tag{1.5}$$

2010 *Mathematics Subject Classification.* 11B68, 11B73, 11B83, 05A40.
Key words and phrases. ordered Bell numbers, ordered Bell polynomials, umbral calculus.

When $x = 0$, $Ch_n = Ch_n(0)$, $(n \geq 0)$ are called the *Changhee numbers*.

In number theory and enumerative combinatorics, the ordered Bell numbers count the number of weak orderings on a set of n elements. The ordered Bell numbers may be computed via a summation formula involving binomial coefficients, or by using a recurrence relation. The ordered Bell numbers appear in the work of Cayley (1859), who used them to count certain plane trees with $n+1$ totally ordered leaves. The nth ordered Bell number may be given by a summation formula involving Stirling numbers of the second kind, which count the number of partitions of an n-element set into k-nonempty sets, expanded out into a double summation involving binomial coefficients, or given by an infinite series (see [1, 2, 4-6, 17]).

The *ordered Bell polynomials* are defined by the generating function

$$\frac{1}{2-e^t}e^{xt} = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}, \text{ (see [2]).} \tag{1.6}$$

When $x = 0$, $b_n = b_n(0)$ are called the *ordered Bell numbers* (see [2, 4]).

From (1.6), we obtain

$$\begin{aligned}\frac{1}{2-e^t} &= \frac{1}{1-(e^t-1)} = \sum_{m=0}^{\infty}(e^t-1)^m \\ &= \sum_{m=0}^{\infty} m! \sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} S_2(n,m)m!\right)\frac{t^n}{n!}.\end{aligned} \tag{1.7}$$

Thus, by (1.6) and (1.7), we get an expression of the ordered Bell numbers in terms of Stirling numbers of the second kind:

$$b_n = \sum_{k=0}^{n} k!S_2(n,k), \ (n \geq 0). \tag{1.8}$$

It is not difficult to show that

$$\begin{aligned}\frac{1}{2-e^t} &= \frac{1}{2}\left(\frac{1}{1-\frac{e^t}{2}}\right) = \frac{1}{2}\sum_{m=0}^{\infty}\left(\frac{e^t}{2}\right)^m \\ &= \frac{1}{2}\sum_{m=0}^{\infty}\left(\frac{1}{2}\right)^m \sum_{n=0}^{\infty} m^n\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2}\sum_{m=0}^{\infty}\frac{m^n}{2^m}\right)\frac{t^n}{n!}.\end{aligned} \tag{1.9}$$

From (1.7), (1.8) and (1.9), we have another expression for the ordered Bell numbers as an infinite series

$$b_n = \frac{1}{2}\sum_{m=0}^{\infty}\frac{m^n}{2^m} = \sum_{m=0}^{n} m!S_2(n,m), \ (n \geq 0). \tag{1.10}$$

Based on a contour integration on this generating function, the ordered Bell numbers can be expressed by the infinite sum as follows:

$$b_n = \frac{n!}{2}\sum_{k=-\infty}^{\infty}(\log 2 + 2\pi ik)^{-(n+1)}, \ (n \geq 1), \tag{1.11}$$

and approximated as

$$b_n \approx \frac{n!}{2(\log 2)^{n+1}}, \ \text{(see [1, 2, 4-6, 17])}. \tag{1.12}$$

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \middle| a_k \in \mathbb{C} \right\}. \tag{1.13}$$

Let $\mathbb{P} = \mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ denotes the action of the linear functional L on the polynomial $p(x)$, and it is known that the vector space operations on $\mathbb{P}^*$ are defined by

$$\langle L + M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle, \ \langle cL|p(x)\rangle = c\langle L|p(x)\rangle,$$

where c is a complex constant (see [3, 13, 15]).

For $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$, we define a linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n, \ \text{for all } n \geq 0, \ \text{(see [12, 15])}. \tag{1.14}$$

Thus, by (1.13) and (1.14), we get

$$\left\langle t^k \middle| x^n \right\rangle = n! \ \delta_{n,k} \ (n \geq 0, \ k \geq 0), \tag{1.15}$$

where $\delta_{n,k}$ is the Kronecker symbol, (see [12, 15]).

For $f_L(t) = \sum_{k=0}^{\infty} \langle L|x^n\rangle \frac{t^k}{k!}$, by (1.15), we get

$$\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle, \ (n \geq 0). \tag{1.16}$$

Additionally, the map $L \mapsto f_L(t)$ is a vector space isomorphism for $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of the formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the umbral algebra.

The umbral calculus is the study of umbral algebra (see [12, 13, 15]). By (1.15), we easily get $\langle e^{yt}| x^n\rangle = y^n$ and so $\langle e^{yt}| p(x)\rangle = p(y)$. The order $o(f(t))$ of a power series $f(t)(\neq 0) \in \mathcal{F}$ is the smallest integer k for which the coefficient of t^k does not vanish (see [12, 15]).

For $f(t), g(t) \in \mathcal{F}$ with $o(f(t)) = 1$ and $o(g(t)) = 0$, there exists a unique sequence $S_n(x)$ of polynomials such that $\left\langle g(t)(f(t))^k \middle| S_n(x)\right\rangle = n!\delta_{n,k}, \ (n, k \geq 0)$. The sequence $S_n(x)$ is called the *Sheffer sequence* for $(g(t), f(t))$ which is denoted by $S_n(x) \sim (g(t), f(t))$ (see [12, 13, 15]). It is well known that $S_n(x) \sim (g(t), f(t))$ if and only if

$$\frac{1}{g((\bar{f}(t))} e^{x\bar{f}(t)} = \sum_{k=0}^{\infty} S_k(x)\frac{t^k}{k!}, \ (x \in \mathbb{C}), \tag{1.17}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ with $f(\bar{f}(t)) = \bar{f}(f(t)) = t$ (see [12, 15]).

For $S_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), l(t))$, $(n \geq 0)$, we have

$$S_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x), \ (n \geq 0), \tag{1.18}$$

where

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} \left(l(\bar{f}(t))\right)^m \middle| x^n \right\rangle, \text{ (see [15]).} \tag{1.19}$$

In this paper, we study the ordered Bell numbers and polynomials and investigate some interesting properties of these numbers and polynomials arising from umbral calculus. In particular, we derive several formulas for the ordered Bell numbers and polynomials, and express well known family of polynomials in terms of the ordered Bell polynomials.

2. On ordered Bell numbers and polynomials

Now, we consider the *ordered Bell polynomials* which are given by the generating function

$$\frac{1}{2-e^t} e^{xt} = \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!}. \tag{2.1}$$

Note that $b_n = b_n(0)$ are the *ordered Bell numbers*. Also, we observe from (2.1) that the ordered Bell polynomials are Appell polynomials and hence in particular $b_n(x)$ has degree n, for each $n \geq 0$. Further, the equation (2.3) in below shows that $b_n(x)$ are monic polynomials with integral coefficients.

From (2.1), we have

$$\begin{aligned} \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!} &= \frac{1}{2-e^t} e^{xt} = \left(\sum_{m=0}^{\infty} b_m \frac{t^m}{m!} \right) \left(\sum_{l=0}^{\infty} x^l \frac{t^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} b_m x^{n-m} \right) \frac{t^n}{n!}. \end{aligned} \tag{2.2}$$

By (1.8) and (2.2), we get

$$\begin{aligned} b_n(x) &= \sum_{m=0}^{n} \binom{n}{m} b_m x^{n-m} \\ &= \sum_{m=0}^{n} \binom{n}{m} \sum_{l=0}^{m} l! S_2(m, l) x^{n-m}. \end{aligned} \tag{2.3}$$

Moreover,

$$\begin{aligned} \sum_{n=0}^{\infty} b_n(x+y) \frac{t^n}{n!} &= \frac{1}{2-e^t} e^{(x+y)t} = \left(\frac{1}{2-e^t} e^{xt} \right) e^{yt} \\ &= \left(\sum_{l=0}^{\infty} b_l(x) \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} \frac{y^m}{m!} t^m \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} b_l(x) y^{n-l} \right) \frac{t^n}{n!}. \end{aligned} \tag{2.4}$$

Thus, by (2.4), we get

$$b_n(x+y) = \sum_{l=0}^{n} \binom{n}{l} b_l(x) y^{n-l}, \ (n \geq 0). \tag{2.5}$$

From (1.5), we can derive the following equation:

$$\begin{aligned} 2\sum_{n=0}^{\infty} b_n \frac{t^n}{n!} &= \frac{2}{2-e^t} = \sum_{m=0}^{\infty} Ch_m \frac{1}{m!}(-e^t)^m \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} (-1)^m Ch_m \frac{m^n}{m!} \right) \frac{t^n}{n!}. \end{aligned} \tag{2.6}$$

Therefore, by (2.6), we obtain the following theorem which expresses the ordered Bell numbers by means of Changhee numbers.

Theorem 2.1. *For $n \geq 0$, we have*

$$b_n = \frac{1}{2} \sum_{m=0}^{\infty} (-1)^m Ch_m \frac{m^n}{m!} = \sum_{m=0}^{\infty} \frac{m^n}{2^{m+1}}.$$

The next gives the connection between the ordered Bell polynomials and Frobenius-Euler polynomials. Indeed, by (1.4), we get

$$\frac{1-2}{e^t-2} e^{xt} = \frac{1}{2-e^t} e^{xt} = \sum_{n=0}^{\infty} H_n(2|x) \frac{t^n}{n!}, \tag{2.7}$$

and

$$\frac{1}{2-e^t} e^{xt} = \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!}. \tag{2.8}$$

By (2.7) and (2.8), we get

$$b_n(x) = H_n(2|x), \ (n \geq 0). \tag{2.9}$$

Here we would like to obtain recurrence relations for the ordered Bell polynomials and numbers from which we can easily compute those polynomials and numbers recursively. From (2.1), we note that

$$\begin{aligned} e^{xt} &= (2-e^t) \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!} \\ &= 2\sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!} - \left(\sum_{m=0}^{\infty} \frac{t^m}{m!} \right) \left(\sum_{l=0}^{\infty} b_l(x) \frac{t^l}{l!} \right) \\ &= 2\sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!} - \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} b_l(x) \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(2b_n(x) - \sum_{l=0}^{n} \binom{n}{l} b_l(x) \right) \frac{t^n}{n!}. \end{aligned} \tag{2.10}$$

Thus, by (2.10), we get

$$x^n = 2b_n(x) - \sum_{l=0}^{n} \binom{n}{l} b_l(x), \ (n \geq 0). \tag{2.11}$$

In particular, $x = 0$, we have

$$2b_n - \sum_{l=0}^{n} \binom{n}{l} b_l = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n \geq 1. \end{cases} \tag{2.12}$$

By (2.12), we get

$$b_0 = 1,\ b_n = \sum_{l=0}^{n-1} \binom{n}{l} b_l,\ (n \geq 1). \tag{2.13}$$

We summarize our results in the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$b_0(x) = 1,\ b_n(x) = x^n + \sum_{l=0}^{n-1} \binom{n}{l} b_l(x),\ (n \geq 1),$$

and

$$b_0 = 1,\ b_n = \sum_{l=0}^{n-1} \binom{n}{l} b_l,\ (n \geq 1).$$

Moreover, $b_n(x) = H_n(2|x),\ (n \geq 0)$.

As an illustration, we compute the first few ordered Bell polynomials from the recurrence relation in the above theorem as follows:

$$\begin{aligned}
&b_0(x) = 1,\ b_1(x) = x + 1,\ b_2(x) = x^2 + 2x + 3,\\
&b_3(x) = x^3 + 3x^2 + 9x + 13,\ b_4(x) = x^4 + 4x^3 + 18x^2 + 52x + 75,\\
&b_5(x) = x^5 + 5x^4 + 30x^3 + 130x^2 + 375x + 541,\\
&b_6(x) = x^6 + 6x^5 + 45x^4 + 260x^3 + 1125x^2 + 3246x + 4683.
\end{aligned}$$

Next, we are interested in expressing some known family of polynomials in terms of the ordered Bell polynomials. For this purpose, we will first obtain Theorem 2.3 which is a general result for this direction. From (1.17) and (2.1), we note that

$$b_n(x) \sim (2 - e^t, t). \tag{2.14}$$

Let

$$\mathbb{P}_n = \{ p(x) \in \mathbb{C}[x] |\deg p(x) \leq n \},\ (n \geq 0). \tag{2.15}$$

Then, it is an $(n + 1)$-dimensional vector space over $\mathbb{C}$. Now, we consider the polynomial $p(x)$ in $\mathbb{P}_n$, which is given by

$$p(x) = \sum_{m=0}^{n} a_m b_m(x),\ (n \geq 0). \tag{2.16}$$

Thus, by (2.16), we get

$$\begin{aligned}
\left\langle (2 - e^t)t^m | p(x) \right\rangle &= \sum_{l=0}^{n} a_l \left\langle (2 - e^t)t^m | b_l(x) \right\rangle \\
&= \sum_{l=0}^{n} a_l l! \delta_{m,l} = m! a_m.
\end{aligned} \tag{2.17}$$

From (2.17), we have

$$a_m = \frac{1}{m!}\left\langle (2-e^t)t^m | p(x) \right\rangle, \ (m \geq 0). \tag{2.18}$$

Therefore, by (2.16) and (2.18), we have the following theorem.

Theorem 2.3. *Let $p(x) \in \mathbb{P}_n$. Then we have*

$$p(x) = \sum_{m=0}^{n} a_m b_m(x), \ (n \geq 0),$$

where $a_m = \frac{1}{m!}\left\langle (2-e^t)t^m | \, p(x) \right\rangle$.

Theorem 2.3 is general enough to be applied to many interesting special polynomials. However, here we will be content with utilizing it for just a few polynomials. Let $p(x) = B_n(x) \in \mathbb{P}_n, \ (n \geq 0)$. Then, by Theorem 2.3, we get

$$B_n(x) = \sum_{m=0}^{n} a_m b_m(x), \ (n \geq 0), \tag{2.19}$$

where

$$\begin{aligned} a_m =& \frac{1}{m!}\left\langle (2-e^t)t^m | B_n(x) \right\rangle = \binom{n}{m} \left\langle (2-e^t) | B_{n-m}(x) \right\rangle \\ =& \binom{n}{m} \left(\langle 2 | B_{n-m}(x) \rangle - \left\langle e^t | B_{n-m}(x) \right\rangle \right) = \binom{n}{m} \left(2B_{n-m} - B_{n-m}(1) \right). \end{aligned} \tag{2.20}$$

Thus, by (2.19), we get

$$\begin{aligned} B_n(x) =& \sum_{m=0}^{n} \binom{n}{m} (2B_{n-m} - B_{n-m}(1)) b_m(x) \\ =& b_n(x) + n(B_1 - 1) b_{n-1}(x) + \sum_{m=0}^{n-2} \binom{n}{m} B_{n-m} b_m(x) \\ =& \sum_{m=0}^{n} \binom{n}{m} B_{n-m} b_m(x) - n b_{n-1}(x). \end{aligned}$$

Hence we obtain the following theorem which expresses the Bernoulli polynomials in terms of the ordered Bell polynomials.

Theorem 2.4. *For $n \geq 0$, we have*

$$B_n(x) = \sum_{m=0}^{n} \binom{n}{m} B_{n-m} b_m(x) - n b_{n-1}(x).$$

For $p(x) = x^n \in \mathbb{P}_n \ (n \geq 0)$, we have

$$x^n = \sum_{k=0}^{n} a_k b_k(x), \tag{2.21}$$

where

$$\begin{aligned} a_k =& \frac{1}{k!}\left\langle (2-e^t)t^k | \, x^n \right\rangle = \binom{n}{k} \left\langle (2-e^t) | \, x^{n-k} \right\rangle \\ =& \binom{n}{k} \left\{ \left\langle 2 | \, x^{n-k} \right\rangle - \left\langle e^t | \, x^{n-k} \right\rangle \right\} = \binom{n}{k} (2\delta_{n,k} - 1). \end{aligned}$$

The following theorem expresses x^n as a linear combination of the ordered Bell polynomials.

Theorem 2.5. *For $n \geq 0$, we have*

$$x^n = \sum_{k=0}^{n} \binom{n}{k} (2\delta_{n,k} - 1) b_k(x).$$

Moreover, $n \geq 1$,

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} \left(S_1(n,k) - \sum_{l=k+1}^{n} \binom{l}{k} S_1(n,l) \right) b_k(x),$$

where $S_1(n,l)$ is the Stirling number of the first kind.

For $p(x) = Bel_n(x) = \sum_{l=0}^{n} S_2(n,l)x^l$, we have

$$Bel_n(x) = \sum_{k=0}^{n} a_k b_k(x), \ (n \geq 0), \tag{2.22}$$

where

$$\begin{aligned} a_k =& \frac{1}{k!} \left\langle (2-e^t)t^k \middle| Bel_n(x) \right\rangle = \frac{1}{k!} \left\langle (2-e^t)t^k \middle| \sum_{l=0}^{n} S_2(n,l)x^l \right\rangle \\ =& \frac{1}{k!} \sum_{l=k}^{n} S_2(n,l)(l)_k \left\langle (2-e^t) \middle| x^{l-k} \right\rangle \\ =& \sum_{l=k}^{n} S_2(n,l) \binom{l}{k} \left\langle 2-e^t \middle| x^{l-k} \right\rangle \\ =& 2\sum_{l=k}^{n} \binom{l}{k} S_2(n,l) \left\langle 1 \middle| x^{l-k} \right\rangle - \sum_{l=k}^{n} S_2(n,l) \binom{l}{k} \left\langle e^t \middle| x^{l-k} \right\rangle \\ =& 2\sum_{l=k}^{n} \binom{l}{k} S_2(n,l)\delta_{l,k} - \sum_{l=k}^{n} S_2(n,l) \binom{l}{k} \\ =& 2S_2(n,k) - \sum_{l=k}^{n} S_2(n,l) \binom{l}{k} \\ =& S_2(n,k) - \sum_{l=k+1}^{n} S_2(n,l) \binom{l}{k}. \end{aligned} \tag{2.23}$$

Therefore, by (2.22) and (2.23), we obtain the following theorem which expresses the (unordered) Bell polynomials in terms of ordered Bell polynomials.

Theorem 2.6. *For $n \geq 0$, we have*

$$Bel_n(x) = \sum_{k=0}^{n} \left\{ S_2(n,k) - \sum_{l=k+1}^{n} S_2(n,l) \binom{l}{k} \right\} b_k(x).$$

For $p(x) = H_n(u|x)$, $(n \geq 0)$, we have

$$H_n(u|x) = \sum_{k=0}^{n} a_k b_k(x), \tag{2.24}$$

where

$$\begin{aligned} a_k =&\frac{1}{k!}\left\langle (2-e^t)t^k \middle| H_n(u|x)\right\rangle = \binom{n}{k}\left\langle (2-e^t) \middle| H_{n-k}(u|x)\right\rangle \\ =&\binom{n}{k}\left\{\langle 2\,|H_{n-k}(u|x)\rangle - \left\langle e^t \middle| H_{n-k}(u|x)\right\rangle\right\} \\ =&\binom{n}{k}\left(2H_{n-k}(u) - H_{n-k}(u|1)\right). \end{aligned} \tag{2.25}$$

Now, we note that

$$H_n(u|1) - uH_n(u) = 0, \text{ for } n > 0, \text{ and } H_0(u) = H_0(u|1) = 1.$$

From (2.25), we have

$$\begin{aligned} H_n(u|x) =&\sum_{k=0}^{n}\binom{n}{k}\left(2H_{n-k}(u) - H_{n-k}(u|1)\right)b_k(x) \\ =&b_n(x) + (2-u)\sum_{k=0}^{n-1}\binom{n}{k}H_{n-k}(u)b_k(x). \end{aligned}$$

Hence we obtain the following theorem which expresses the Frobenius-Euler polynomials as a linear combination of the ordered Bell polynomials.

Theorem 2.7. *For $n \geq 0$, we have*

$$H_n(u|x) = b_n(x) + (2-u)\sum_{k=0}^{n-1}\binom{n}{k}H_{n-k}(u)b_k(x).$$

Remark 1.

$$\frac{1}{2-e^t}e^{xt} = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}.$$

Thus we note that

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{d}{dx}b_n(x)\frac{t^n}{n!} =&\frac{\partial}{\partial x}\left(\frac{1}{2-e^t}e^{xt}\right) = \frac{t}{2-e^t}e^{xt} \\ =&t\sum_{n=0}^{\infty}b_n(x)\frac{t^n}{n!} = \sum_{n=1}^{\infty}nb_{n-1}(x)\frac{t^n}{n!}. \end{aligned} \tag{2.26}$$

By (2.26), we get

$$\frac{d}{dx}b_n(x) = nb_{n-1}(x), \ (n \geq 1). \tag{2.27}$$

This time we will use (1.18) and (1.19) instead of Theorem 2.3 in order to express the (unordered) Bell polynomials in terms of the ordered Bell polynomials. Let us consider the following Sheffer sequences:

$$Bel_n(x) \sim (1, \log(1+t)), \ b_n(x) \sim (2-e^t, t).$$

Then, by (1.18) and (1.19), we get

$$Bel_n(x) = \sum_{m=0}^{n} C_{n,m}b_m(x), \tag{2.28}$$

where

$$\begin{aligned} C_{n,m} =&\frac{1}{m!}\left\langle \left(2-e^{(e^t-1)}\right)(e^t-1)^m \middle| x^n\right\rangle \\ =&\sum_{l=m}^{n}\frac{S_2(l,m)}{l!}\left\langle \left(2-e^{(e^t-1)}\right)t^l \middle| x^n\right\rangle \\ =&\sum_{l=m}^{n}\frac{S_2(l,m)}{l!}\left\{2\left\langle t^l\middle| x^n\right\rangle - \left\langle e^{(e^t-1)}t^l \middle| x^n\right\rangle\right\} \\ =&2S_2(n,m)-\sum_{l=m}^{n}\binom{n}{l}S_2(l,m)\left\langle e^{e^t-1}\middle| x^{n-l}\right\rangle. \end{aligned} \tag{2.29}$$

Now, we observe that

$$\begin{aligned} \left\langle e^{e^t-1}\middle| x^{n-l}\right\rangle =&\sum_{k=0}^{\infty}\frac{Bel_k(1)}{k!}\left\langle t^k\middle| x^{n-l}\right\rangle \\ =&\sum_{k=0}^{n-l}\frac{Bel_k(1)}{k!}\left\langle t^k\middle| x^{n-l}\right\rangle \\ =&\sum_{k=0}^{n-l}\frac{Bel_k(1)}{k!}(n-l)!\delta_{n-l,k} \\ =&Bel_{n-l}(1). \end{aligned} \tag{2.30}$$

From (2.29) and (2.30), we note that

$$\begin{aligned} C_{n,m} =&2S_2(n,m)-\sum_{l=m}^{n}\binom{n}{l}S_2(l,m)Bel_{n-l}(1) \\ =&S_2(n,m)-\sum_{l=m}^{n-1}\binom{n}{l}S_2(l,m)Bel_{n-l}(1). \end{aligned} \tag{2.31}$$

Therefore, by (2.28) and (2.31), we obtain the following theorem.

Theorem 2.8. *For $n \geq 0$, we have*

$$Bel_n(x)=\sum_{m=0}^{n}\left\{S_2(n,m)-\sum_{l=m}^{n-1}\binom{n}{l}S_2(l,m)Bel_{n-l}(1)\right\}b_m(x).$$

Remark 2. By comparing Theorems 2.6 and 2.8, we immediately get the following identity:

$$\sum_{l=k}^{n-1}\binom{n}{l}S_2(l,k)Bel_{n-l}(1)=\sum_{l=k+1}^{n}\binom{l}{k}S_2(n,l),\ (0\leq k\leq n-1).$$

Finally, we would like to express the ordered Bell polynomials as a linear combination of falling factorial polynomials. For $b_n(x)\sim(2-e^t,t)$, $(x)_n\sim(1,e^t-1)$, we have

$$b_n(x)=\sum_{m=0}^{n}C_{n,m}(x)_m, \tag{2.32}$$

where

$$\begin{aligned} C_{n,m} =& \frac{1}{m!}\left\langle \frac{1}{2-e^t}(e^t-1)^m \middle| x^n \right\rangle \\ =& \sum_{l=m}^{\infty} S_2(l,m)\frac{1}{l!}\left\langle \frac{1}{2-e^t}t^l \middle| x^n \right\rangle \\ =& \sum_{l=m}^{n} S_2(l,m)\binom{n}{l}\left\langle \frac{1}{2-e^t} \middle| x^{n-l} \right\rangle \\ =& \sum_{l=m}^{n} S_2(l,m)\binom{n}{l} b_{n-l}. \end{aligned} \tag{2.33}$$

Therefore, by (2.32) and (2.33), we obtain the following theorem.

Theorem 2.9. *For $n \geq 0$, we have*

$$b_n(x) = \sum_{m=0}^{n}\left(\sum_{l=m}^{n} S_2(l,m)\binom{n}{l} b_{n-l}\right)(x)_m.$$

References

[1] A. Cayley, *On the analytical forms called trees, second part*, Philosophical Magazine, Series IV, **18** (121) (1859), 374–378.

[2] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, D. Reidel Publishing Co., 1974, p. 228.

[3] R. Dere and Y. Simsek, *Applications of umbral algebra to some special polynomials*, Adv. Stud. Contemp. Math. (Kyungshang), **22**, (2012), no. 3, 433-438.

[4] J. Good, *The number of orderings of n candidates when ties are permitted*, Fibonacci Quart., **13**, (1975), 11-18.

[5] O. A. Gross, *Preferential arrangements*, Amer. Math. Monthly, **69** (1962), 4-8.

[6] J. Harris, J. L. Hirst and M. J. Mossinghoff, *Combinatorics and Graph Theory. Undergraduate Texts in Mathematics (2nd ed.)*, Springer, 2008, p. 132.

[7] L. C. Jang and T. Kim, *Some identities of Bell polynomials*, J. Comput. Anal. Appl., **20** (2016), no. 3, 584-589.

[8] D. Kang, S. J. Lee, J. W. Park and S. H. Rim, *On the twisted weak weight q-Bernoulli polynomials and numbers*, Proc. Jangjeon Math. Soc., **16** (2013), no. 2, 195-201.

[9] B. M. Kim, J. Jeong and S. H. Rim *Some explicit identities on Changhee-Genocchi polynomials and numbers*, Adv. Difference Equ. 2016, **2016**:202, 12 pp.

[10] D. S. Kim and T. Kim, *Some identities of Bell polynomials*, Sic. China. Math., **58**, (2015), no. 10, 2095-2104.

[11] T. Kim, *An identity of the symmetry for the Frobenius-Euler polynomials associated with the fermionic p-adic invariant q-integrals on $\mathbb{Z}_p$*, Rocky Mountain J. Math., **41**, (2011), no. 1, 239-247.

[12] T. Kim, *Identities involving Laguerre polynomials derived from umbral calculus*, Russ. J. Math. Phys., **21** (2014), no. 1, 36-45.

[13] T. Kim and T. Mansour, *Umbral calculus associated with Frobenius-type Eulerian polynomials*, Russ. J. Math. Phys., **21** (2014), no. 4, 484-493.

[14] J. W. Park, *On the twisted q-Changhee polynomials of higher order*, J. Comput. Anal. Appl., **20** (2016), no. 3, 424-431.

[15] S. Roman, *The umbral calculus*, Pure and Applied Mathematics, vol. 111, Academic Press, New York, 1984.

[16] Y. Simsek, *Generating functions of the twisted Bernoulli numbers and polynomials associated with their interpolation functions*, Adv. Stud. Contemp. Math. (Kyungsang), **16** (2008), no. 2, 251-278.

[17] A. Sklar, *On the factorization of sqare free integers*, Proc. Amer. Math. Soc., **3** (1952), 701-705.

[18] Y. Yu and Y. He, *Some recurrence relations for q-Bernoulli numbers and polynomials* Ars Combin., **127** (2016), 157-165.

Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
E-mail address: tkkim@kw.ac.kr

Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea
E-mail address: dskim@sogang.ac.kr

Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
E-mail address: gwjang@kw.ac.kr